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Note

Remarks on the size of critical edge-chromatic graphs

Lane H. Clark *, Dawit Haile

Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, IL 62901, USA

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Abstract

We give new lower bounds for the size of Δ-critical edge-chromatic graphs when $6 \leqslant \Delta \leqslant 21$.

1. Introduction

All graphs we consider are undirected and have neither loops nor multiple edges. We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. The order of G is $|V(G)|$ and the size of G is $|E(G)|$. We denote the degree of a vertex v in G by $d_G(v)$ and the maximum degree of G by $\Delta(G)$. A vertex of maximum degree is called a major vertex; otherwise it is a minor vertex. The number of vertices in G of degree k is denoted by $n_k = n_k(G)$. The subgraph of G induced by a subset A of vertices of G is denoted by $G[A]$. For disjoint subsets A, B of vertices of G, $E(A,B)$ denotes the set of edges one end of which is in A and the other end of which is in B. We write $e(A,B)$ for $|E(A,B)|$ and $e(v,B)$ for $e(\{v\},B)$. Our notation and terminology generally follow [1].

The chromatic index $\chi'(G)$ of a graph G is the minimum number of colors required to color the edges of G so that no two adjacent edges receive the same color. Vizing [7] showed that $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G)+1$. A graph G is of class 1 provided that $\chi'(G) = \Delta(G)$, and G is of class 2 provided that $\chi'(G) = \Delta(G)+1$. We say G is Δ-critical if and only if $\Delta(G) = \Delta$, G is connected, G is of class 2 and $\chi'(G-e) < \chi'(G)$ for every edge e of G.

For a Δ-critical graph G of order n and size m, Vizing [7] conjectured that $m \geqslant \frac{1}{2}[(\Delta - 1)n + 3)]$. Jakobsen [5] showed that $m \geqslant \frac{4}{3}n$ for $\Delta = 3$, thus verifying

* Corresponding author.

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the conjecture. Fiorini and Wilson [3] showed that

$$m \geqslant \begin{cases} \frac{5}{3}n & \text{if } \Delta = 4, \\ \frac{9}{5}n & \text{if } \Delta = 5, \\ 2n & \text{if } \Delta = 6, \end{cases}$$

thus verifying the conjecture for $\Delta = 4$. Yap [8] further improved these results by showing

$$m \geqslant \begin{cases} 2n+1 & \text{if } \Delta = 5, \\ \frac{1}{4}(9n+1) & \text{if } \Delta = 6, \\ \frac{5}{2}n & \text{if } \Delta = 7. \end{cases}$$

Recently, Kayathri [6] improved the result of Yap [8] by showing $m \geqslant 2n+2$ for $\Delta = 5$, thus verifying the conjecture. In general, the best bounds are due to Fiorini [2] who showed that

$$m \geqslant \begin{cases} \frac{1}{4}(\Delta+1)n & \text{if } \Delta \text{ is odd}, \\ \frac{1}{4}(\Delta+2)n & \text{if } \Delta \text{ is even}. \end{cases}$$

In this paper we further improve the bounds of Yap [8] and Fiorini [2] with the following result.

Theorem. *Let G be a Δ-critical graph of order n and size m. For $6 \leqslant \Delta \leqslant 21$, we have*

$$m \geqslant f(\Delta)n,$$

where

$$f(\Delta) = \begin{cases} \dfrac{\Delta+1}{3} & \text{for } 6 \leqslant \Delta \leqslant 8, \\ \dfrac{\Delta+4}{4} & \text{for } 9 \leqslant \Delta \leqslant 12, \\ \dfrac{3\Delta+20}{14} & \text{for } 13 \leqslant \Delta \leqslant 16, \\ \dfrac{3\Delta+30}{16} & \text{for } 17 \leqslant \Delta \leqslant 21. \end{cases}$$

We require the following theorems.

Theorem 1 (Fiorini and Wilson [3], and Yap [9]). *Let G be a Δ-critical graph and let $vw \in E(G)$ where $d_G(v) = k$. Then*

(i) *if $k < \Delta$ then w is adjacent to at least $\Delta - k + 1$ major vertices of G,*
(ii) *if $k = \Delta$ then w is adjacent to at least two major vertices of G,*
(iii) $d_G(v) + d_G(w) \geqslant \Delta + 2$.

The result in (i) and (ii) of Theorem 1 is known as the Vizing Adjacency Lemma (VAL).

Theorem 2 (Fiorini and Wilson [3] and Yab [9]). *A critical graph contains no cut vertex. Moreover, there are no regular Δ-critical graphs for $\Delta \geqslant 3$.*

2. Main result

We now fix a Δ-critical graph G with $\Delta \geqslant 3$. Let

$$A_k = \{v \in V(G)\colon d_G(v) = k\} \quad \text{for } 2 \leqslant k \leqslant \Delta;\ |A_k| = n_k.$$

$$A_{k,l} = \{v \in A_k\colon e(v, A_\Delta) = l\} \quad \text{for } 2 \leqslant l \leqslant k \leqslant \Delta - 1;\ |A_{k,l}| = a_{k,l}.$$

For $2 \leqslant k \leqslant \Delta - 1$, VAL(i) and (ii) give

$$\sum_{l=2}^{k} a_{k,l} = n_k.$$

Now we may restate VAL(i) in terms of the notation introduced above.

Lemma 3. *If $v \in A_k$ with $2 \leqslant k \leqslant \Delta - 1$ and $vw \in E(G)$, then w is adjacent to at least $\Delta - k + 1$ major vertices, hence, $w \in A_\Delta \cup \bigcup_{\Delta-1 \geqslant q > r \geqslant \Delta-k+1} A_{q,r}$.*

Although the following result can be read out of the proof of [3, Theorem 13.5], we include its proof for completeness. See [4, Corollary 3.2.6] for an entirely different proof.

Lemma 4. *We have*

$$n_\Delta \geqslant \sum_{k=2}^{\Delta-1} \sum_{l=2}^{k} \frac{l a_{k,l}}{k-1}. \tag{1}$$

Proof. For $v \in A_\Delta$, consider $(v_2, \ldots, v_{\Delta-1})$ where $v_k = e(v, A_k)$ for $2 \leqslant k \leqslant \Delta - 1$. Let $T = \{(v_2, \ldots, v_{\Delta-1})\colon v \in A_\Delta\}$; $T^* = T - \{(0, \ldots, 0)\}$; $B(i_2, \ldots, i_{\Delta-1}) = \{v \in A_\Delta\colon (v_2, \ldots, v_{\Delta-1}) = (i_2, \ldots, i_{\Delta-1})\}$ and $b(i_2, \ldots, i_{\Delta-1}) = |B(i_2, \ldots, i_{\Delta-1})| (\neq 0)$ for $(i_2, \ldots, i_{\Delta-1}) \in T$. Observe that $\{B(i_2, \ldots, i_{\Delta-1})\colon (i_2, \ldots, i_{\Delta-1}) \in T\}$ partitions A_Δ. For each $2 \leqslant k \leqslant \Delta - 1$,

$$\sum_{l=2}^{k} l a_{k,l} = e(A_\Delta, A_k) = \sum_{(i_2, \ldots, i_{\Delta-1}) \in T^*} i_k b(i_2, \ldots, i_{\Delta-1})$$

hence,

$$\sum_{k=2}^{\Delta-1}\sum_{l=2}^{k}\frac{la_{k,l}}{k-1}=\sum_{k=2}^{\Delta-1}\sum_{(i_2,\ldots,i_{\Delta-1})\in T^*}\frac{i_k b(i_2,\ldots,i_{\Delta-1})}{k-1}$$
$$=\sum_{(i_2,\ldots,i_{\Delta-1})\in T^*} b(i_2,\ldots,i_{\Delta-1})\sum_{k=2}^{\Delta-1}\frac{i_k}{k-1}.$$

Fix $(i_2,\ldots,i_{\Delta-1})\in T^*$, and let $q=q(i_2,\ldots,i_{\Delta-1})=\min\{k\colon i_k\neq 0\}$, so that, $2\leqslant q\leqslant \Delta-1$. Observe that for $v\in B(i_2,\ldots,i_{\Delta-1})$ with $i_q\neq 0$, there exists $vw\in E(G)$ with $w\in A_q$. By VAL(i), v is adjacent to at least $\Delta-q+1$ major vertices, so, at most $q-1$ minor vertices. Hence, $i_q+\cdots+i_{\Delta-1}=i_2+\cdots+i_{\Delta-1}\leqslant q-1$. Consequently,

$$\sum_{k=2}^{\Delta-1}\sum_{l=2}^{k}\frac{la_{k,l}}{k-1}\leqslant\sum_{(i_2,\ldots,i_{\Delta-1})\in T^*} b(i_2,\ldots,i_{\Delta-1})\sum_{k=q=q(i_2,\ldots,i_{\Delta-1})}^{\Delta-1}\frac{i_k}{q-1}$$
$$\leqslant\sum_{(i_2,\ldots,i_{\Delta-1})\in T^*} b(i_2,\ldots,i_{\Delta-1})\leqslant n_\Delta. \qquad\square$$

Theorem 5. *For $\Delta\geqslant 5$, we have*

$$n_\Delta\geqslant 2n_2+3\sum_{k=3}^{\Delta-2}\frac{n_k}{k-1}+\frac{n_{\Delta-1}}{\Delta-3}. \tag{2}$$

Proof. Let $v\in A_{k,2}$ and $vw\in E(G[V-A_\Delta])$. Lemma 3 implies $w\in A_{q,r}$ with $\Delta-1\geqslant q>r\geqslant\Delta-k+1$. Now, Lemma 3 implies $v\in A_{s,t}$ with $\Delta-1\geqslant s>t\geqslant\Delta-q+1$. Since $t=2$, $w\in\bigcup_{\Delta-2\geqslant r\geqslant\Delta-k+1}A_{\Delta-1,r}$, hence, $A_{k,2}$ is independent for $2\leqslant k\leqslant\Delta-2$. Thus, for each $3\leqslant k\leqslant\Delta-2$, we have

$$(k-2)a_{k,2}=e(A_{k,2},A_{\Delta-1})$$
$$=e\left(A_{k,2},\bigcup_{\Delta-2\geqslant r\geqslant\Delta-k+1}A_{\Delta-1,r}\right)\leqslant\sum_{l=2}^{k-1}(l-1)a_{\Delta-1,\Delta-l}.$$

Hence,

$$\sum_{k=3}^{\Delta-2}\frac{a_{k,2}}{k-1}=\sum_{k=3}^{\Delta-2}\frac{k-2}{(k-1)(k-2)}a_{k,2}$$
$$\leqslant\sum_{k=3}^{\Delta-2}\frac{1}{(k-1)(k-2)}\sum_{l=2}^{k-1}(l-1)a_{\Delta-1,\Delta-l}$$
$$=\sum_{l=2}^{\Delta-3}(l-1)a_{\Delta-1,\Delta-l}\sum_{k=l}^{\Delta-3}\frac{1}{k(k-1)}$$

$$= \sum_{l=2}^{\Delta-3}(l-1)a_{\Delta-1,\Delta-l}\sum_{k=l}^{\Delta-3}\left[\frac{1}{k-1}-\frac{1}{k}\right]$$

$$= \sum_{l=2}^{\Delta-3}\frac{\Delta-l-2}{\Delta-3}a_{\Delta-1,\Delta-l}. \tag{3}$$

For $\Delta-3 \geqslant l \geqslant 2$, we have

$$\frac{\Delta-l}{\Delta-2} \geqslant \frac{\Delta-l-2}{\Delta-3}+\frac{1}{\Delta-3}. \tag{4}$$

Using (3) and (4) in (1), we obtain

$$\begin{aligned}
n_\Delta &\geqslant 2n_2+\sum_{k=3}^{\Delta-2}\sum_{l=2}^{k}\frac{la_{k,l}}{k-1}+\sum_{l=2}^{\Delta-1}\frac{la_{\Delta-1,l}}{\Delta-2}\\
&= 2n_2+\sum_{k=3}^{\Delta-2}\sum_{l=2}^{k}\frac{la_{k,l}}{k-1}+\sum_{l=2}^{\Delta-3}\frac{(\Delta-l)a_{\Delta-1,\Delta-l}}{\Delta-2}+\sum_{l\in\{2,\Delta-1\}}\frac{la_{\Delta-1,l}}{\Delta-2}\\
&\geqslant 2n_2+\sum_{k=3}^{\Delta-2}\sum_{l=2}^{k}\frac{la_{k,l}}{k-1}+\sum_{l=2}^{\Delta-3}\frac{(\Delta-l-2)a_{\Delta-1,\Delta-l}}{\Delta-3}\\
&\quad +\sum_{l=2}^{\Delta-3}\frac{a_{\Delta-1,\Delta-l}}{\Delta-3}+\sum_{l\in\{2,\Delta-1\}}\frac{la_{\Delta-1,l}}{\Delta-2}\\
&\geqslant 2n_2+\sum_{k=3}^{\Delta-2}\sum_{l=3}^{k}\frac{la_{k,l}}{k-1}+3\sum_{k=3}^{\Delta-2}\frac{a_{k,2}}{k-1}+\sum_{l=2}^{\Delta-1}\frac{a_{\Delta-1,l}}{\Delta-3}\\
&\geqslant 2n_2+3\sum_{k=3}^{\Delta-2}\sum_{l=2}^{k}\frac{a_{k,l}}{k-1}+\sum_{l=2}^{\Delta-1}\frac{a_{\Delta-1,l}}{\Delta-3}\\
&= 2n_2+3\sum_{k=3}^{\Delta-2}\frac{n_k}{k-1}+\frac{n_{\Delta-1}}{\Delta-3}. \qquad \square
\end{aligned}$$

Lemma 6. *Let $f_2(c), f_3(c), \ldots, f_{\Delta-1}(c)$ be positive, decreasing linear functions on $[0,\Delta]$ and $f_\Delta(c)=c$. Set $g(c)=\min_{2\leqslant k\leqslant\Delta}\{f_k(c)\}$ for $c\in[0,\Delta]$. Then g is continuous on $[0,\Delta]$ and*

$$\max_{0\leqslant c\leqslant\Delta} g(c) = \min_{2\leqslant k\leqslant\Delta-1}\{c\colon f_k(c)=f_\Delta(c)\}. \tag{5}$$

Proof. Clearly g is continuous on $[0,\Delta]$ as it is a polygonal function. Let $f_k(c)=f_\Delta(c)$ at $c_k\in(0,\Delta]$. If necessary, reindex so that $c_2\leqslant c_3\leqslant\cdots\leqslant c_{\Delta-1}$ (this does *not* change the value of g). Then $g(c)=f_\Delta(c)$ on $[0,c_2]$ and $g(c)\leqslant f_2(c)$ on $[c_2,\Delta]$. Since f_Δ is an increasing function and f_2 is a decreasing function on $[0,\Delta]$,

$$\max_{0\leqslant c\leqslant\Delta} g(c)=g(c_2)=c_2=\min_{2\leqslant k\leqslant\Delta-1}\{c\colon f_k(c)=f_\Delta(c)\}. \qquad \square$$

We now give our main result.

Theorem 7. *Let G be a Δ-critical graph of order n and size m. For $6 \leqslant \Delta \leqslant 21$, we have*

$$m \geqslant f(\Delta)n$$

where

$$f(\Delta) = \begin{cases} \dfrac{\Delta+1}{3} & \text{for } 6 \leqslant \Delta \leqslant 8, \\ \dfrac{\Delta+4}{4} & \text{for } 9 \leqslant \Delta \leqslant 12, \\ \dfrac{3\Delta+20}{14} & \text{for } 13 \leqslant \Delta \leqslant 16, \\ \dfrac{3\Delta+30}{16} & \text{for } 17 \leqslant \Delta \leqslant 21. \end{cases}$$

Proof. From (2) we see that

$$\begin{aligned} 2m &= \sum_{k=2}^{\Delta} kn_k \\ &= 2n_2 + \sum_{k=3}^{\Delta-1} kn_k + cn_\Delta + (\Delta - c)n_\Delta \\ &\geqslant 2n_2 + \sum_{k=3}^{\Delta-1} kn_k + cn_\Delta + (\Delta - c)\left[2n_2 + 3\sum_{k=3}^{\Delta-2} \frac{n_k}{k-1} + \frac{n_{\Delta-1}}{\Delta-3}\right] \\ &= (2\Delta + 2 - 2c)n_2 + \sum_{k=3}^{\Delta-2} \frac{k^2 - k + 3\Delta - 3c}{k-1} n_k \\ &\quad + \frac{\Delta^2 - 3\Delta + 3 - c}{\Delta - 3} n_{\Delta-1} + cn_\Delta \end{aligned} \tag{6}$$

for any constant $c \in [0, \Delta]$. Let $f_2(c) = 2\Delta + 2 - 2c$, $f_k(c) = (k^2 - k + 3\Delta - 3c)/(k-1)$ $(3 \leqslant k \leqslant \Delta - 2)$, $f_{\Delta-1}(c) = (\Delta^2 - 3\Delta + 3 - c)/(\Delta - 3)$ and $f_\Delta(c) = c$. Letting $f_k(c) = c$ at c_k, we obtain $c_2 = (2\Delta + 2)/3$, $c_k = (k^2 - k + 3\Delta)/(k+2)$ $(3 \leqslant k \leqslant \Delta - 2)$ and $c_{\Delta-1} = (\Delta^2 - 3\Delta + 3)/(\Delta - 2)$. Observe that $c_2 < c_{\Delta-1}$ for all real Δ. From (5), the optimal value of c is then obtained by comparing c_2 with $c_3, \ldots, c_{\Delta-2}$. Analysis of each $\Delta \in \{6, \ldots, 21\}$ gives $c = c_2 = (2\Delta + 2)/3$ for $6 \leqslant \Delta \leqslant 8$; $c = c_4 = (\Delta + 4)/2$ for $9 \leqslant \Delta \leqslant 12$; $c = c_5 = (3\Delta + 20)/7$ for $13 \leqslant \Delta \leqslant 16$; and $c = c_6 = (3\Delta + 30)/8$ for $17 \leqslant \Delta \leqslant 21$. Our result follows upon using these optimal values in (6). □

Remark 8. Since vertices of degree 2 are adjacent only to major vertices, the term $2n_2$ in (2) cannot, in general, be improved. For $6 \leqslant \Delta \leqslant 8$, the optimal value of c came from $f_2(c) = f_\Delta(c)$. Consequently, our result there is best possible given our approach.

For $\Delta \in \{9,10,11,12\}$, $\{13,14,15,16\}$ or $\{17,18,19,20,21\}$, our results likely are not best possible since optimal values of c came from $f_k(c) = f_\Delta(c)$ with $k = 4, 5$, and 6, respectively. Finally, we note that (6) also gives a new bound for $\Delta = 23$.

3. Conclusion

For $6 \leqslant \Delta \leqslant 21$, we have given new lower bounds for the size of a Δ-critical graph of order n. Moreover, our result is best possible given our approach for $6 \leqslant \Delta \leqslant 8$. Unfortunately, the method we have employed gives no information for $\Delta \geqslant 24$. We hope to come back to the general case in a future paper.

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